Lecture Note 1 on

Optimisation and Energy Pricing

Module of Advanced Power Systems

ELECTENG 703SC

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Te Whare Wānanga o Tāmaki Makaurau

SYSTEM OPERATOR

- Linear Algebra
- Transformations of similar representations
- Hyperplanes and Half Spaces, convex function
- Polyhedra and polytopes
- Representation of Linear Programming problems (LPs)
- Solution of LP by SIMPLEX algorithm
- Degeneracy and Cycling
- Dual LP – Properties and Interpretations
- Kuhn-Tucker Conditions and Duality

Linear algebra (LA)

- Basic tool for understanding Linear Programming Problem (LPP)
- Main components of LA are matrices and vectors
- Basic operations on matrices:
  - transpose: (A^T);
  - inverse: (A^{-1})
  - The rank(A) = No. of independent rows or columns of A

$$Ax = b$$

$$x = A^{-1}b$$

- Properties of Transpose

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T; \text{ Note order change}$$

$$(\mathrm{rA})^{\mathrm{T}} = rA^T$$

- Properties of Inverse

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}; \text{ Note order change}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$AA^{-1} = I_n$$

- Linear combination: Vector v is linear combination of vectors v1, v2, …vk if

$$v = c_1v_1 + c_2v_2 + c_3v_3 + ... + c_kv_k$$

- Linearly dependent: Set of distinct vectors, S={v1,v2…vk}. S is said to be Linearly dependent, if

$$c_1, c_2, c_3 \ldots . c_k \text{ not all zero such that}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots . + c_k v_k = 0 \ldots \ldots (1)$$

Otherwise S is Linearly independent, i.e, if (1) can be satisfied only with

$$c_1 = c_2 = c_3 \ldots . = c_k = 0$$

The essential point in the above definition is whether the eqn (1) can hold with not all of the constants c1, c2, …ck being zero.

Optimization objectives

Two possibilities:

- minimization, and
- maximization

$$\min f(x) = -\max - f(x)$$

However,

Thus, we can study just one of them

- Representing LPs

  – as a system of inequalities
  – as a system of equalities
- Change of equality to inequality

$$A_{i*}.x = b_i$$
$$\Rightarrow A_{i*}.x \le b_i \text{ and } A_{i*}.x \ge b_i$$
$$A_{i*} = i^{th} \text{ row of } A$$

- Slack Variable

Slack variables are important, since they allow us to convert from inequalities to equalities.

$$A_{i*}.x \le b_i \quad \Rightarrow \quad A_{i*}.x + s = b_i$$

Where $s \ge 0$ is a non-negative slack variable

- Reversing an inequality

$$k_1 x_1 + k_2 x_2 + \ldots . + k_n x_n \geq b$$

Multiply by -1 on both sides

$$-k_1 x_1 - k_2 x_2 - \ldots . - k_n x_n \leq -b$$

Non-negativity of variables

- Non-negativity can be enforced by inequalities, for any vector x:

$$x \geq 0$$

- Unrestricted variables can be represented as a sum of nonnegative variables:

$$x = x^{+} - x^{-}$$

where x is unrestricted and x^{+} and x^{-} are non-negative

- Hyperplane

A hyperplane H in R^n is the set

$$H = \{x \in R^n \mid c^T x = k\}$$

$$ex.\ ax + by = k; merely\ a\ line$$

H divides R^n into two subsets:

$$H_1 = \{x \in R^n \mid c^T x \le k\}; and$$

$$H_2 = \{x \in R^n \mid c^T x \ge k\}$$

H_1 and H_2 $closed\ half-spaces$

$$Open\ half\ spaces$$

$$H_1^* = \{x \in R^n \mid c^T x < k\};$$

$$H_2^* = \{x \in R^n \mid c^T x > k\}; and$$

Convex Set

- A subset S of R^n is called convex if for any two distinct points x1 and x2 in S the line segment joining x1 and x2 lies in S. That is, S is convex if whenever $x_1, x_2 \in S$ so does

$$x=\lambda x_1+(1-\lambda)x_2;\text{for } 0\le\lambda\le 1$$

Convex function

A function f is defined on a convex set in R^n is called a convex function if

$$f[\lambda x_1+(1-\lambda)x_2]\le\lambda f(x_1)+(1-\lambda)f(x_2);\text{for } 0\le\lambda\le 1$$

and $x_1, x_2 \in S.$

- A closed/open space is a convex set
- A hyperplane is a convex set
- The intersection of a finite collection of convex set is convex
- Polyhedron and Polytopes

A polyhedron is the intersection of a finite number of half spaces.
It is clear that a polyhedron is also a convex set.

Polyhedron can be

- bounded
- unbounded

A polytope is a bounded polyhedron.

- In a convex polytope, every point is a convex combination of extreme points

What are linear models?

$$\text{Maximize} \quad z = \mathbf{c}^T\mathbf{x} \tag{1}$$

subject to

$$\mathbf{Ax} \leq \mathbf{b} \tag{2}$$

$$\mathbf{x} \geq \mathbf{0} \tag{3}$$

A(mxn); x(nx1); b(mx1), c(nx1)

nsform each of the constraints in (2) into an equation by lack variable. We obtain the canonical form of the problem

$$\text{Maximize} \quad z = \mathbf{c}^T\mathbf{x} \tag{4}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{5}$$

$$\mathbf{x} \geq \mathbf{0} \tag{6}$$

ase **A** is the $m \times (n + m)$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Basic Solution, Feasible Solution, Basic Feasible Solution and Optimal Solution

Basic Solution: A solution to $Ax \geq b$, it does not necessarily satisfy $x \geq 0$

Non-Basic and Basic Variables: In any basic solution, the (s-m) variables which are set to zero are called Non-Basic variables and the m variables solved for are called Basic Variables.[s=n+m variables, m=no of constraints and each constraint has a slack variable].

Feasible solution: Solution that satisfy the constraints of a LP is a Feasible solution

Basic Feasible Solution: Is a Basic Solution which is also feasible.

Optimal solution: A Feasible solution which minimises or maximises the objective function of a LPP as called an Optimal solution

EXAMPLE 1. **(Activity analysis or product mix)** A lumber mill saws both construction-grade and finish-grade boards from the logs which it receives. Suppose that it takes 2 hours to rough-saw each 1000 board feet of the construction-grade boards and 3 hours to plane each 1000 board feet of these boards. Suppose that it also takes 2 hours to rough-saw each 1000 board feet of the finish-grade boards, but it takes 5 hours to plane each 1000 board feet of these boards. The saw is available 8 hours per day and the plane is available 15 hours per day. If the profit on each 1000 board feet of construction-grade boards is \$100 and the profit on each 1000 board feet of finish-grade boards is \$120, how many board feet of each type of lumber should be sawed to maximize the profit?

Mathematical Model. Let x and y denote the amount of fi and construction-grade lumber, respectively, to be sawed pe the units of x and y be thousands of board feet. The numbe required daily for the saw is

$$2x + 2y$$

Since only 8 hours are available daily, x and y must satisfy the

$$2x + 2y \le 8$$

Similarly, the number of hours required for the plane is

$$5x + 3y$$

so x and y must satisfy

$$5x + 3y \le 15$$

Thus, our mathematical model is

Find values of x and y which will

$$\text{maximize} \quad z = 120x + 100y$$

subject to the restrictions

$$\begin{aligned} 2x + 2y &\le 8 \\ 5x + 3y &\le 15 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

helpful to examine it geometrically.

Consider a linear programming problem in standard form

$$\text{Maximize} \quad z = \mathbf{c}^T\mathbf{x} \tag{1}$$

subject to

$$\mathbf{Ax} \leq \mathbf{b} \tag{2}$$

$$\mathbf{x} \geq \mathbf{0} \tag{3}$$

Section 2.2. In this form it is:

$$\text{Maximize} \quad z = 120x + 100y \qquad (7$$

subject to

$$\left.\begin{aligned} 2x + 2y + u \quad\quad &= 8 \\ 5x + 3y \quad\quad + v &= 15 \end{aligned}\right\} \qquad (8$$

$$x \geq 0, \quad y \geq 0, \quad u \geq 0, \quad v \geq 0 \qquad (9$$

Tableau 2.1

	x	y	u	v	z	
u	2	2	1	0	0	8
v	5	3	0	1	0	15
	−120	−100	0	0	1	0

Tableau 2.1a

↓

	x	y	u	v	z	
u	2	2	1	0	0	8
← v	5	3	0	1	0	15
	−120	−100	0	0	1	0

Tableau 2.3a

	x	y	u	v	z	
u	0	$\frac{4}{5}$	1	$-\frac{2}{5}$	0	2
x	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	3
	0	-28	0	24	1	360

Tableau 2.4

	x	y	u	v	z	
y	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	0	$\frac{5}{2}$
x	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	0	$\frac{3}{2}$
	0	0	35	10	1	430

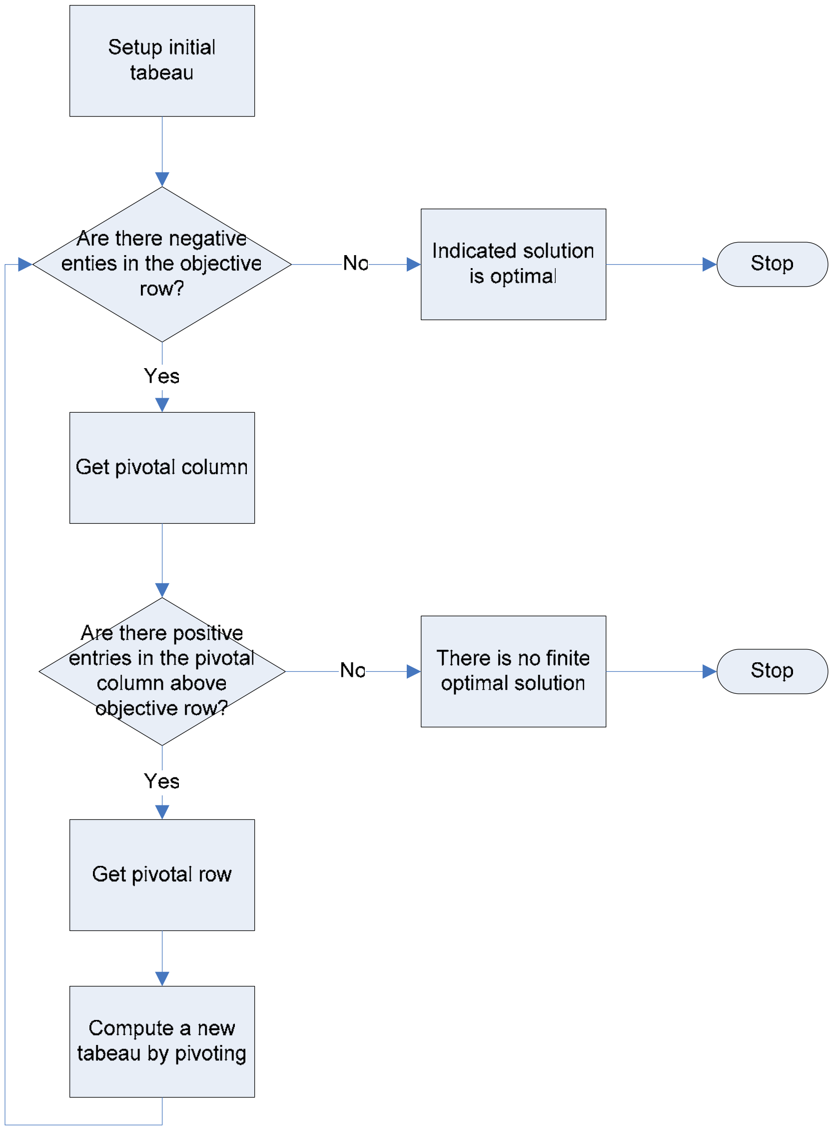
Optimality Criterion

If the objective row of a tableau has

- Zero entries in the columns labelled by basic variables, and

-No-negative entries in the columns labelled by non-basic variables,

- then the solution represented by the tableau is optimal.



Flowchart for simplex algorithm (standard form $b \geq 0$)

Geometric Interpretation

Table 2.3

Extreme point (x, y)	Value of objective function $z = 120x + 100y$
(0, 0)	0
(3, 0)	360
(0, 4)	400
$(\frac{3}{2}, \frac{5}{2})$	430

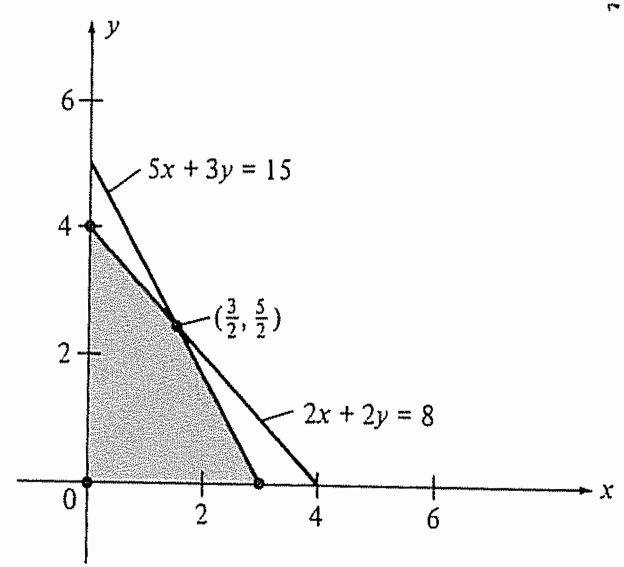


Figure 2.2

Degeneracy

Tableau 2.9

	x_1	x_2	$\cdots$	x_j	$\cdots$	x_{n+m}	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_j	a_{r1}/a_{rj}	a_{r2}/a_{rj}	$\cdots$	1	$\cdots$	$a_{r,n+m}/a_{rj}$	b_r/a_{rj}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{i_s}	*	*	$\cdots$	0	$\cdots$	*	$(b_s - a_{sj} \cdot b_r)/a_{rj}$

indicates an entry whose value we are not concerned about. Setting the nonbasic variables in Tableau 2.9 equal to zero, we find that

$$x_j = b_r/a_{rj}$$

and

$$x_{i_s} = b_s - a_{sj} \cdot b_r/a_{rj} = a_{sj}(b_s/a_{sj} - b_r/a_{rj}) = 0$$

Consequently, the tie among the θ-ratios has produced a basic variable whose value is 0.

Definition. A basic feasible solution in which some basic variables are zero is called **degenerate**.

Maximize $\quad z = 5x_1 + 3x_2$

subject to

$$
\begin{aligned}
x_1 - x_2 &\le 2 \\
2x_1 + x_2 &\le 4 \\
-3x_1 + 2x_2 &\le 6 \\
x_1 \ge 0, \quad x_2 &\ge 0
\end{aligned}
$$

Degeneracy

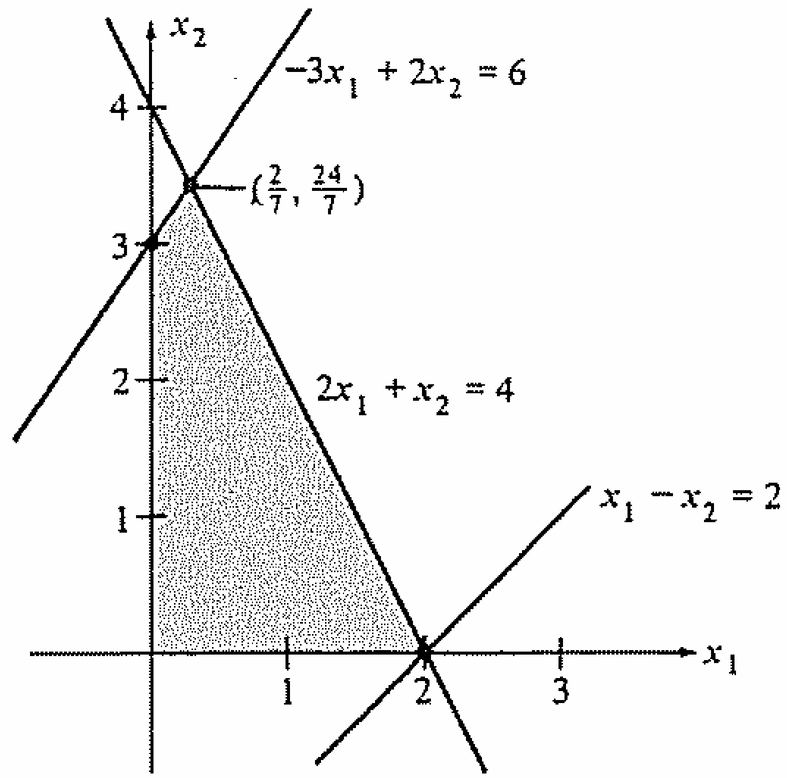


Table 2.6

Extreme point	Value of $z = 5x_1 + 3x_2$
(0, 0)	0
(2, 0)	10
(0, 3)	9
$(\frac{2}{7}, \frac{24}{7})$	$\frac{82}{7}$

Tableau 2.10

	x_1	x_2	x_3	x_4	x_5	
x_3	1	−1	1	0	0	2
x_4	2	1	0	1	0	4
x_5	−3	2	0	0	1	6
	−5	−3	0	0	0	0

Tableau 2.11

	x_1	x_2	x_3	x_4	x_5	
x_1	1	−1	1	0	0	2
x_4	0	3	−2	1	0	0
x_5	0	−1	3	0	1	12
	0	−8	5	0	0	10

Tableau 2.12

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	2
x_2	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	0
x_5	0	0	$\frac{7}{3}$	$\frac{1}{3}$	1	12
	0	0	$-\frac{1}{3}$	$\frac{8}{3}$	0	10

Tableau 2.13

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$
x_2	0	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{24}{7}$
x_3	0	0	1	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{36}{7}$
	0	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{82}{7}$

Tableau 2.10a

	↓ x_1	x_2	x_3	x_4	x_5	
x_3	1	−1	1	0	0	2
← x_4	2	1	0	1	0	4
x_5	−3	2	0	0	1	6
	−5	−3	0	0	0	0

Tableau 2.11a

	x_1	↓ x_2	x_3	x_4	x_5	
x_3	0	$-\frac{3}{2}$	1	$-\frac{1}{2}$	0	0
x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	2
← x_5	0	$\frac{7}{2}$	0	$\frac{3}{2}$	1	12
	0	$-\frac{1}{2}$	0	$\frac{5}{2}$	0	10

Tableau 2.12a

	x_1	x_2	x_3	x_4	x_5	
x_3	0	0	1	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{36}{7}$
x_1	1	0	0	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$
x_2	0	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{24}{7}$
	0	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{82}{7}$

with the optimal value of the objective function as

$$z = \tfrac{82}{7}$$

The slack variables have values

$$x_3 = \tfrac{36}{7}, \quad x_4 = 0, \quad x_5 = 0$$

What is happening geometrically? We start with the initial basic feasible solution as the origin (0, 0), where $z = 0$. If we choose to replace x_3 with x_1, we move to the adjacent extreme point (2, 0) where $z = 10$ (Tableau 2.11). Now we replace x_4 by x_2 and remain at (2, 0) (Tableau 2.12). Finally we replace x_5 with x_3 and move to $(\frac{2}{7}, \frac{24}{7})$, where $z = \frac{82}{7}$. This is our optimal solution (Tableau 2.13).

If instead we choose to replace x_4 with x_1 at our point of degeneracy, we again move to (2, 0), where $z = 10$ (Tableau 211a). However, at the next stage, x_3, which has value 0 and is the degenerate variable, is not a departing variable. Instead, x_5 is the departing variable, and we move immediately to the optimal solution (Tableau 2.12a) at $(\frac{2}{7}, \frac{24}{7})$.

Duality

$$\left.\begin{array}{l}\text{Maximize} \quad z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ \qquad \mathbf{A}\mathbf{x} \le \mathbf{b} \\ \qquad \;\; \mathbf{x} \ge \mathbf{0}\end{array}\right\} \qquad (1)$$

$$\left.\begin{array}{l}\text{Minimize} \quad z' = \mathbf{b}^T\mathbf{w} \\ \text{subject to} \\ \qquad \mathbf{A}^T\mathbf{w} \ge \mathbf{c} \\ \qquad \;\; \mathbf{w} \ge \mathbf{0}\end{array}\right\}$$

EXAMPLE 1. If the primal problem is

$$\text{Maximize} \quad z = [2 \quad 3]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 3 & 2 \\ -1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$x_1 \ge 0, \quad x_2 \ge 0$$

then the dual problem is

$$\text{Minimize} \quad z' = [2 \quad 5 \quad 1]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \ge \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$w_1 \ge 0, \quad w_2 \ge 0, \quad w_3 \ge 0$$

Example Chair (X1)

Table(X2)

Primal problem: Max

$$\Pi = X_1 + X_2$$

$$5X_1 + X_2 \leq 200$$

$$X_1 + 2X_2 \leq 90$$ Labour

$$X_1, X_2 \geq 0$$ Raw material

Its dual problem : Min

$$V = 200W_1 + 90W_2$$

$$5W_1 + W_2 \geq 1$$

$$W_1 + 2W_2 \geq 1$$ Chair

$$W_1, W_2 \geq 0$$ Table

Theorem 3.4. If $\mathbf{x}_0$ is a feasible solution to the primal problem

$$\left.\begin{array}{l} \text{Maximize} \quad z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \\ \qquad \mathbf{A}\mathbf{x} \le \mathbf{b} \\ \qquad \mathbf{x} \ge \mathbf{0} \end{array}\right\} \qquad (8)$$

and $\mathbf{w}_0$ is a feasible solution to the dual problem

$$\left.\begin{array}{l} \text{Minimize} \quad z' = \mathbf{b}^T\mathbf{w} \\ \text{subject to} \\ \qquad \mathbf{A}^T\mathbf{w} \ge \mathbf{c} \\ \qquad \mathbf{w} \ge \mathbf{0} \end{array}\right\} \qquad (9)$$

then

$$\mathbf{c}^T\mathbf{x}_0 \le \mathbf{b}^T\mathbf{w}_0$$

That is, the value of the objective function of the dual problem is
greater than or equal to the value of the objective function of the
problem.

Theorem 3.5. If $\mathbf{x}_0$ and $\mathbf{w}_0$ are feasible solutions to the primal and dual problems (8) and (9), respectively, and $\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{w}_0$, then both $\mathbf{x}_0$ and $\mathbf{w}_0$ are optimal solutions to their respective problems.

Duality and the Kuhn-Tucker Optimality Conditions

Recall from Chapter 5 that the optimality conditions for a linear program state that a necessary and sufficient condition for $\mathbf{x}^*$ to be an optimal point to the linear program Minimize $\mathbf{cx}$ subject to $\mathbf{Ax} > \mathbf{b}$, $\mathbf{x} > \mathbf{0}$ is that there exists a vector

$\mathbf{w}^*$ such that

$$
\begin{aligned}
&1.\ \mathbf{Ax}^* > \mathbf{b},\ \mathbf{x}^* > \mathbf{0}\\
&2.\ \mathbf{w}^*\mathbf{A} < \mathbf{c},\ \mathbf{w}^* > \mathbf{0}\\
&3.\ \mathbf{w}^*(\mathbf{Ax}^* - \mathbf{b}) = 0\\
&\quad\ (\mathbf{c} - \mathbf{w}^*\mathbf{A})\mathbf{x}^* = 0
\end{aligned}
$$

Condition 1 above simply requires that the optimal point $\mathbf{x}^*$ must be feasible to the primal. In light of our discussion of duality we can now interpret condition 2. This condition indicates that the vector $\mathbf{w}^*$ must be a feasible point for the dual problem. From condition 3 above, we find that $\mathbf{cx}^* = \mathbf{w}^*\mathbf{b}$. Hence $\mathbf{w}^*$ must be an optimal solution to the dual problem. The Kuhn-Tucker optimality conditions for the dual problem imply the existence of a primal feasible solution whose objective is equal to that of the optimal dual (why?). This leads to the following lemma.

Kuhn-Tucker Conditions and Duality

Primal problem

$$MIN\ Z_P = f(x)$$

subject to:

$$h_i(x) = c_i; i = 1, 2, \ldots I$$

$$g_j(x) \le b_j;\ j = 1, 2, \ldots J$$

$$h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \\ \\ h_I(x) \end{bmatrix};\ g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ \\ g_J(x) \end{bmatrix}$$

x is a vector of state variables, control variables and parameters.

If $x = (x_1, x_2, x_3 \ldots . x_n) \in R^n$ is minimiser, there there exist

$\lambda \in R^I$ and $\mu \in R^J$

We define Lagrangian Function:

$$L = f(x) + \lambda^T h(x) + \mu^T g(x)$$

KKT conditions are:

$$\frac{\partial f}{\partial x} + \lambda^T \frac{\partial h}{\partial x} + \mu^T \frac{\partial h}{\partial x} = 0$$

$$h(x) = 0 \quad (h(x) = c_i \text{ for } i = 1, 2, \ldots I)$$

$$g(x) \leq 0 \quad (g_j(x) \leq b_j \text{ for j} = 1, 2, \ldots J)$$

$$\mu_j.(g_j(x) - b_j) = 0 \text{ for j} = 1, 2, \ldots J$$

$$\mu_j \geq 0 \text{ for j} = 1, 2, \ldots J$$

$$\begin{bmatrix} \mu_j.(g_j(x) - b_j) = 0 \\ \mu_j \geq 0 \end{bmatrix} = j^{th} \text{complementary slackness condition}$$

If $\mu_j > 0;$ then $g_j(x) = b_j;$ i.e. constrant is binding that means all the j th

resources are exausted at the optimal solution and needs more.

If $\mu_j = 0;$ then $g_j(x) \leq b_j;$ i.e. constrant is NOT binding.

- The objective function of Dual problem is Lagrangian type of function formed from equality and inequality constraints. At the optimal solution the elements in vector x assumes the optimal values and thus the term f(x) in primal can be treated as constant and so can be disregarded in Dual maximisation problem.

- The remaining terms are complementary slackness conditions as discussed above.

- Any price λ, μ that satisfy both the dual constraints and complementary slackness conditions are satisfied, the dual objective function is redundant, If dual solution is not degenerate. Thus prices can be evaluated from the dual constraints which are the Ist order optimality conditions.